

有理関数の積分 (部分分数分解)

1 次の不定積分を求めよ。

$$(1) \int \frac{dx}{(x+1)(x+2)^2}$$

【解答】 $\frac{1}{(x+1)(x+2)^2} = \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}$ とおいて

$$a=1, b=-1, c=-1$$

$$\begin{aligned} \int \frac{dx}{(x+1)(x+2)^2} &= \int \left\{ \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2} \right\} dx \\ &= \log|x+1| - \log|x+2| + \frac{1}{x+2} + C = \log \left| \frac{x+1}{x+2} \right| + \frac{1}{x+2} + C \end{aligned}$$

$$(2) \int \frac{x+1}{x^3-1} dx$$

【解答】 $\frac{x+1}{x^3-1} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}$ とおいて

$$a = \frac{2}{3}, b = -\frac{2}{3}, c = -\frac{1}{3}$$

$$\begin{aligned} \int \frac{x+1}{x^3-1} dx &= \frac{2}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{2x+1}{x^2+x+1} dx \\ &= \frac{2}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{(x^2+x+1)'}{x^2+x+1} dx \\ &= \frac{2}{3} \log|x-1| - \frac{1}{3} \log(x^2+x+1) + C = \frac{1}{3} \log \frac{(x-1)^2}{x^2+x+1} + C \end{aligned}$$

$$(3) \int \frac{x}{(x+1)(x^2+1)} dx$$

【解答】 $\frac{x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$ とおいて

$$a = -\frac{1}{2}, b = \frac{1}{2}, c = \frac{1}{2}$$

$$\begin{aligned} \frac{x}{(x+1)(x^2+1)} &= -\frac{1}{2(x+1)} + \frac{x+1}{2(x^2+1)} \\ \int \frac{x}{(x+1)(x^2+1)} dx &= -\frac{1}{2} \int \frac{1}{x+1} + \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log(x^2+1) + \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

2 次の不定積分を求めよ。

$$(1) \int \frac{dx}{(x+a)(x+b)} \quad (a \neq b)$$

【解答】

(1)

$$\begin{aligned} \frac{1}{(x+a)(x+b)} &= \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right) \text{ であるから} \\ \int \frac{dx}{(x+a)(x+b)} &= \int \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right) dx \\ &= \frac{1}{b-a} \int \left(\frac{1}{x+a} - \frac{1}{x+b} \right) dx \\ &= \frac{1}{b-a} (\log|x+a| - \log|x+b|) + C \\ &= \frac{1}{b-a} \log \left| \frac{x+a}{x+b} \right| + C \end{aligned}$$

$$(2) \int \frac{dx}{x^2-a^2}$$

【解答】

$$\begin{aligned} \frac{1}{x^2-a^2} &= \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) \text{ であるから} \\ \int \frac{dx}{x^2-a^2} &= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx \\ &= \frac{1}{2a} (\log|x-a| - \log|x+a|) + C \\ &= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \end{aligned}$$

$$(3) \int \frac{1}{(x^2+a^2)(x^2+b^2)} dx \quad (a \neq 0, b \neq 0)$$

【解答】

$$\begin{aligned} \frac{1}{(x^2+a^2)(x^2+b^2)} &= \frac{1}{b^2-a^2} \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2} \right) \text{ であるから} \\ \int \frac{1}{(x^2+a^2)(x^2+b^2)} dx &= \frac{1}{b^2-a^2} \int \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2} \right) dx \\ &= \frac{1}{b^2-a^2} \left(\frac{1}{a} \tan^{-1} \frac{x}{a} - \frac{1}{b} \tan^{-1} \frac{x}{b} \right) + C \end{aligned}$$

有理関数の積分 (部分分数分解)

3 次の不定積分を求めよ。

$$(1) \int \frac{x}{x^4 + x^2 + 1} dx$$

【解答】基本事項 $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$ ($a \neq 0$)

$$\int \frac{lx + m}{x^2 + 2px + q} dx = \log(x^2 + 2px + q) + \frac{m - lp}{\sqrt{q - p^2}} \tan^{-1} \frac{x + p}{\sqrt{q - p^2}} + C \quad (q > p^2)$$

$$\frac{x}{x^4 + x^2 + 1} = \frac{x}{(x^2 + x + 1)(x^2 - x + 1)} = \frac{ax + b}{x^2 + x + 1} + \frac{cx + d}{x^2 - x + 1} \quad \text{とおいて}$$

$$a = 0, b = -\frac{1}{2}, c = 0, d = \frac{1}{2}$$

$$\int \frac{x}{x^4 + x^2 + 1} dx = \frac{1}{2} \int \frac{dx}{(x - \frac{1}{2})^2 + \frac{3}{4}} - \frac{1}{2} \int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}} \left(\tan^{-1} \frac{2x - 1}{\sqrt{3}} - \tan^{-1} \frac{2x + 1}{\sqrt{3}} \right) + C = -\frac{1}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}}{2x^2 + 1} + C$$

$$(2) \int \frac{1 - x^2}{x(x^4 + x^2 + 1)} dx$$

【解答】 $\frac{1 - x^2}{x(x^4 + x^2 + 1)} = \frac{1}{x} - \frac{x - 1}{2(x^2 + x + 1)} - \frac{x + 1}{2(x^2 - x + 1)}$ として計算できる。

しかし、次のようにするほうが簡単である。

$\frac{1}{x^2} = t$ とおくと $-2 \log|x| = \log t$ 両辺を微分して $\frac{dx}{x} = -\frac{dt}{2t}$ を得る。

$$\int \frac{1 - x^2}{x(x^4 + x^2 + 1)} dx = \int \frac{1 - x^2}{x^4 + x^2 + 1} \cdot \frac{dx}{x}$$

$$= \int \frac{1 - t^{-1}}{t^{-2} + t^{-1} + 1} \left(-\frac{dt}{t} \right) = -\frac{1}{2} \int \frac{t - 1}{t^2 + t + 1} dt$$

$$= -\frac{1}{4} \int \frac{2t + 1}{t^2 + t + 1} dt + \frac{3}{4} \int \frac{dt}{t^2 + t + 1}$$

$$= -\frac{1}{4} \log(t^2 + t + 1) + \frac{\sqrt{3}}{2} \tan^{-1} \frac{2t + 1}{\sqrt{3}} + C$$

$$= \frac{1}{4} \log \frac{x^4}{x^2 + x + 1} + \frac{\sqrt{3}}{2} \tan^{-1} \frac{x^2 + 2}{\sqrt{3}x^2} + C$$

$$(3) \int \frac{dx}{x(a + bx^n)^2}$$

【解答】 $a + bx^n = t$ とおけば $nbx^{n-1} dx = dt$ したがって $\frac{dx}{x} = \frac{dt}{nbx^n} = \frac{1}{n} \cdot \frac{dt}{t - a}$ を代入して

$$\int \frac{dx}{x(a + bx^n)^2} = \frac{1}{n} \int \frac{dt}{t^2(t - a)} = \frac{1}{na^2} \int \left(\frac{1}{t - a} - \frac{1}{t} - \frac{a}{t^2} \right) dt$$

$$= \frac{1}{na^2} \left(\log \left| \frac{bx^n}{a + bx^n} \right| + \frac{a}{a + bx^n} \right) + C$$

$$(4) \int \frac{dx}{x^3 + 1}$$

【解答】 $\frac{1}{x^3 + 1} = \frac{1}{(x + 1)(x^2 - x + 1)} = \frac{a}{x + 1} + \frac{bx + c}{x^2 - x + 1}$ とおけば

$$a = \frac{1}{3}, b = -\frac{1}{3}, c = \frac{2}{3}$$

$$\int \frac{dx}{x^3 + 1} = \frac{1}{3} \int \frac{dx}{x + 1} - \frac{1}{3} \int \frac{x - 2}{x^2 - x + 1} dx$$

$$= \frac{1}{3} \int \frac{dx}{x + 1} - \frac{1}{6} \int \frac{2x - 1}{x^2 - x + 1} dx + \frac{1}{2} \int \frac{dx}{x^2 - x + 1}$$

$$= \frac{1}{3} \log|x + 1| - \frac{1}{6} \log(x^2 - x + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}} + C \quad \text{ここまで}$$

$$= \frac{1}{6} \log \frac{(x + 1)^2}{x^2 - x + 1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}} + C$$

$$= \frac{1}{6} \log \frac{(x + 1)^3}{x^3 + 1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}} + C$$

$$= \frac{1}{2} \log|x + 1| - \frac{1}{6} \log|x^3 + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}} + C$$

$$(5) \int \frac{dx}{x^4 + 1}$$

【解答】 $x^4 + 1 = (x^2 + 1)^2 - 2x^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ より

$$\frac{1}{x^4 + 1} = \frac{ax + b}{x^2 + \sqrt{2}x + 1} + \frac{cx + d}{x^2 - \sqrt{2}x + 1} \quad \text{とおくと } a = \frac{1}{2\sqrt{2}}, b = \frac{1}{2}, c = -\frac{1}{2\sqrt{2}}, d = \frac{1}{2}$$

$$\frac{1}{x^4 + 1} = \frac{1}{2\sqrt{2}} \left(\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right)$$

$$\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} = \frac{x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{x^2 + \sqrt{2}x + 1} = \frac{1}{2} \cdot \frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} + \frac{1}{\sqrt{2}} \cdot \frac{1}{x^2 + \sqrt{2}x + 1}$$

$$\frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} = \frac{x - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{x^2 - \sqrt{2}x + 1} = \frac{1}{2} \cdot \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} - \frac{1}{\sqrt{2}} \cdot \frac{1}{x^2 - \sqrt{2}x + 1} \quad \text{より}$$

$$\int \frac{dx}{x^4 + 1} = \frac{1}{4\sqrt{2}} \log \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} + \frac{1}{2\sqrt{2}} \left\{ \tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1) \right\} + C$$

$$= \frac{1}{4\sqrt{2}} \log \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{1 - x^2} + C$$

(注) $\tan^{-1} m_1 = \theta_1, \tan^{-1} m_2 = \theta_2$ のとき, $\tan \theta_1 = m_1, \tan \theta_2 = m_2$

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

$$\therefore \theta_1 + \theta_2 = \tan^{-1} \left(\frac{m_1 + m_2}{1 - m_1 m_2} \right)$$

$$\therefore \tan^{-1} m_1 + \tan^{-1} m_2 = \tan^{-1} \left(\frac{m_1 + m_2}{1 - m_1 m_2} \right)$$

$$\text{同様に, } \tan^{-1} m_1 - \tan^{-1} m_2 = \tan^{-1} \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right)$$

$$\tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1) = \tan^{-1} \frac{(\sqrt{2}x + 1) + (\sqrt{2}x - 1)}{1 - (\sqrt{2}x + 1)(\sqrt{2}x - 1)} = \tan^{-1} \frac{\sqrt{2}x}{1 - x^2}$$

有理関数の積分 (部分分数分解)

4 次の不定積分を求めよ。

$$(1) \int \frac{lx + (m - lp)}{x^2 + a^2} dx$$

【解答】

$$\int \frac{lx + (m - lp)}{x^2 + a^2} dx = \int \frac{lx}{x^2 + a^2} dx + (m - lp) \int \frac{dx}{x^2 + a^2}$$

$$\text{ここで, } \int \frac{lx}{x^2 + a^2} dx = l \int \frac{x}{x^2 + a^2} dx = \frac{l}{2} \log(x^2 + a^2) + C_1$$

$$\begin{aligned} (m - lp) \int \frac{dx}{x^2 + a^2} &= (m - lp) \int \frac{dx}{a^2 \left\{ \left(\frac{x}{a}\right)^2 + 1 \right\}} \\ &= \frac{m - lp}{a} \tan^{-1} \frac{x}{a} + C_2 \end{aligned}$$

$$\therefore \int \frac{lx + (m - lp)}{x^2 + a^2} dx = \frac{l}{2} \log(x^2 + a^2) + \frac{m - lp}{a} \tan^{-1} \frac{x}{a} + C$$

$$(2) I = \int \frac{dx}{ax^2 + 2bx + c} \quad (a \neq 0)$$

$$\text{【解答】 基本事項 } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C \quad (a \neq 0)$$

$$I = a \int \frac{dx}{(ax + b)^2 + (ac - b^2)} \text{ で}$$

$$ax + b = t \text{ とおけば } dx = \frac{dt}{a} \text{ だから}$$

$$I = \int \frac{dx}{t^2 + (ac - b^2)} \text{ である。}$$

$ac - b^2 > 0$ のときは

$$\begin{aligned} I &= \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{t}{\sqrt{ac - b^2}} + C \\ &= \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{ax + b}{\sqrt{ac - b^2}} + C \end{aligned}$$

$ac - b^2 = 0$ ならば

$$I = -\frac{1}{t} + C = -\frac{1}{ax + b} + C$$

$ac - b^2 < 0$ ならば

$$\begin{aligned} I &= \frac{1}{2\sqrt{b^2 - ac}} \log \left| \frac{t - \sqrt{b^2 - ac}}{t + \sqrt{b^2 - ac}} \right| + C \\ &= \frac{1}{2\sqrt{b^2 - ac}} \log \left| \frac{ax + b - \sqrt{b^2 - ac}}{ax + b + \sqrt{b^2 - ac}} \right| + C \end{aligned}$$

$$(3) I = \int \frac{Lx + M}{x^2 + 2px^2 + q} dx \quad (q - p^2 > 0)$$

【解答】

$$x^2 + 2px^2 + q = (x + p)^2 + q - p^2 \text{ だから } x + p = z, \quad q - p^2 = a^2 \text{ とおけば}$$

$$Lx + M = L(x + p) + (M - Lp), \quad dx = dz \text{ だから}$$

$$\begin{aligned} I &= \int \frac{Lz + (M - Lp)}{z^2 + a^2} dz \\ &= \frac{L}{2} \log(z^2 + a^2) + \frac{M - Lp}{a} \tan^{-1} \frac{z}{a} + C \\ &= \frac{L}{2} \log(x^2 + 2px^2 + q) + \frac{M - Lp}{\sqrt{q - p^2}} \tan^{-1} \frac{x + p}{\sqrt{q - p^2}} + C \end{aligned}$$